

# Approximation of SPDEs with Hölder Continuous Drifts

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## Abstract

In this paper, exploiting the regularities of the corresponding Kolmogorov equations involved we investigate strong convergence of exponential integrator scheme for a range of stochastic partial differential equations, in which the drift term is Hölder continuous, and reveal the rate of convergence.

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## 1 Introduction

The numerical approximation of stochastic partial differential equations (SPDEs) has been a very active field of research. Due to the infinite dimensional nature of the driving noise processes, in order to be able to simulate a numerical approximation on a computer, both temporal discretization and spatial discretization are implemented. The temporal discretization is achieved generally by Euler type approximations, Milstein type approximations, and splitting-up method (see, e.g., [2, 3, 5, 7, 8, 10, 16, 19]), and the spatial discretization is in general done by finite element, finite difference and spectral Galerkin methods (see, e.g., [11, 15]). Also, there are some alternative schemes (for example, stochastic Taylor expansion [12]) to approximate SPDEs. In contrast to substantial literature on approximations of SPDEs with regular coefficients, the counterpart for SPDEs with irregular terms (e.g., Hölder continuous drifts) is scarce. Whereas, our goal in this paper is to make an attempt to discuss strong convergence of an exponential integrator (EI) scheme, coupled with a Galerkin

scheme for the spatial discretization (see (1.8) and (1.9) below), for a class of SPDEs with Hölder continuous drifts. With regard to convergence of EI scheme for SPDEs with smooth drift coefficients, we refer to [13, 14, 17, 18] for further details, to name a few. Also, there is a number of literature on approximation of SPDEs with non-globally Lipschitz continuous nonlinearities; see, for instance, [6, 9].

Let  $(\mathbb{H}, \langle \cdot, \cdot \rangle_{\mathbb{H}}, \|\cdot\|_{\mathbb{H}})$  be a real separable Hilbert space. Denote by  $\mathcal{L}(\mathbb{H})$  (resp.  $\mathcal{L}_2(\mathbb{H})$ ) the space of all bounded linear operators (resp. Hilbert-Schmidt operators) on  $\mathbb{H}$ . Let  $\|\cdot\|$  (resp.  $\|\cdot\|_{\mathcal{L}_2}$ ) stand for the operator norm (resp. the Hilbert-Schmidt norm). Let  $(W_t)_{t \geq 0}$  be an  $\mathbb{H}$ -valued cylindrical Wiener process defined on a complete probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ , i.e.,  $W_t = \sum_{k=1}^{\infty} \beta_t^{(k)} e_k$ , where  $(\beta^{(k)})_{k \geq 1}$  is a sequence of independent real-valued Brownian motions on the probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  and  $(e_k)_{k \geq 1}$  is an orthonormal basis of  $\mathbb{H}$ . Fix  $T > 0$  and set  $\|f\|_{T, \infty} := \sup_{t \in [0, T], x \in \mathbb{H}} \|f(t, x)\|$  for an operator-valued map  $f$  on  $[0, T] \times \mathbb{H}$ . Let  $\mathcal{B}_b(\mathbb{H}; \mathbb{H})$  be the collection of all bounded measurable functions  $f : \mathbb{H} \rightarrow \mathbb{H}$ .

Consider the following semi-linear SPDE on  $\mathbb{H}$

$$(1.1) \quad dX_t = \{AX_t + b_t(X_t)\}dt + dW_t, \quad t \in [0, T], \quad X_0 = x \in \mathbb{H},$$

where

**(A1)**  $(-A, \mathcal{D}(-A))$  is a positive definite self-adjoint operator on  $\mathbb{H}$  having discrete spectrum with all eigenvalues  $(0 <) \lambda_1 \leq \lambda_2 \leq \dots$  counting multiplicities such that

$$(1.2) \quad \sum_{i=1}^{\infty} \frac{1}{\lambda_i^{1-\alpha}} < \infty$$

for some  $\alpha \in (0, 1)$ .

**(A2)**  $b : [0, T] \times \mathbb{H} \rightarrow \mathbb{H}$  is uniformly bounded (i.e.,  $\|b\|_{T, \infty} < \infty$ ) and there exist  $c, \beta > 0$  and  $\varepsilon \in (0, 1)$  with  $\frac{2\beta}{2-\varepsilon} \geq 1 - \alpha$  such that

$$(1.3) \quad \|b_t(x) - b_t(y)\|_{\mathbb{H}} \leq c \sum_{i=1}^{\infty} \frac{1}{\lambda_i^{\beta}} |x_i - y_i|^{\varepsilon}, \quad t \in [0, T], \quad x, y \in \mathbb{H},$$

where  $x_i := \langle x, e_i \rangle$  and  $y_i := \langle y, e_i \rangle$ .

**(A3)** For  $c > 0$  and  $\varepsilon \in (0, 1)$  in (1.3),

$$(1.4) \quad \|b_s(x) - b_t(x)\|_{\mathbb{H}} \leq c|s - t|^{\varepsilon}, \quad s, t \in [0, T], x \in \mathbb{H}.$$

It is easy to see that (1.3) is equivalent to that  $b_t$  is continuous and that

$$(1.5) \quad \|b_t(x) - b_t(x + \langle y - x, e_i \rangle e_i)\|_{\mathbb{H}} \leq \frac{c}{\lambda_i^{\beta}} |x_i - y_i|^{\varepsilon}, \quad t \in [0, T], \quad x, y \in \mathbb{H}.$$

By the Hölder inequality, along with (1.2) and  $\frac{2\beta}{2-\varepsilon} \geq 1 - \alpha$ , (1.3) implies that

$$(1.6) \quad \|b_t(x) - b_t(y)\|_{\mathbb{H}} \leq c_0 \|x - y\|_{\mathbb{H}}^{\varepsilon}, \quad t \in [0, T], \quad x, y \in \mathbb{H}$$

for some  $c_0 > 0$ . That is,  $b_t$  is Hölder continuous of order  $\varepsilon$  w.r.t. the spatial variable. Thus, according to [20, Theorem 1.1], (1.1) has a unique mild solution, i.e., there exists a unique continuous adapted process  $(X_t)_{t \geq 0}$  such that  $\mathbb{P}$ -a.s.,

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A}b_s(X_s)ds + \int_0^t e^{(t-s)A}dW_s, \quad t > 0.$$

For any  $n \in \mathbb{N}$ , let  $\pi_n : \mathbb{H} \mapsto \mathbb{H}_n := \text{span}\{e_1, \dots, e_n\}$  be the orthogonal projection,  $A_n = \pi_n A$ ,  $b_t^{(n)} = \pi_n b_t$  and  $W_t^{(n)} = \pi_n W_t$ . With the notation above, we consider the following finite-dimensional approximation associated with (1.1) on  $\mathbb{H}_n \simeq \mathbb{R}^n$

$$(1.7) \quad dX_t^{(n)} = \{A_n X_t^{(n)} + b_t^{(n)}(X_t^{(n)})\}dt + dW_t^{(n)}, \quad t > 0, \quad X_0^{(n)} = x_n := \pi_n x,$$

which is the Galerkin projection of (1.1) onto  $\mathbb{H}_n$ . Since  $A_n x = Ax$  for any  $x \in \mathbb{H}_n$  and  $b_n$  is Hölder continuous in terms of (1.5), by virtue of [20, Theorem 1.1], (1.7) has a unique strong solution.

Now we define numerical schemes to approximate  $X_t^{(n)}$  in time, which are called discrete-time EI scheme: for a stepsize  $\delta \in (0, 1)$  and each integer  $k \geq 0$ ,

$$(1.8) \quad \bar{Y}_{(k+1)\delta}^{(n),\delta} = e^{\delta A_n} \{\bar{Y}_{k\delta}^{(n),\delta} + b_{k\delta}^{(n)}(\bar{Y}_{k\delta}^{(n),\delta})\delta + \Delta W_k^{(n)}\}, \quad \bar{Y}_0^{(n),\delta} = x_n,$$

which is also named as Lord-Rougemont scheme (see, e.g., [13, (4.5)] and [17, (3.2)]), where  $\Delta W_k^{(n)} := W_{(k+1)\delta}^{(n)} - W_{k\delta}^{(n)}$ , and continuous-time EI scheme

$$(1.9) \quad Y_t^{(n),\delta} = e^{tA_n}x_n + \int_0^t e^{(t-s\delta)A_n}b_{s\delta}^{(n)}(Y_{s\delta}^{(n),\delta})ds + \int_0^t e^{(t-s\delta)A_n}dW_s^{(n)}, \quad t \geq 0,$$

where  $t_\delta := \lfloor t/\delta \rfloor \delta$  with  $\lfloor t/\delta \rfloor$  being the integer part of  $t/\delta$ . It is easy to see that  $Y_{k\delta}^{(n),\delta} = \bar{Y}_{k\delta}^{(n),\delta}$  for any  $k \geq 0$ .

The main result of this paper is stated as follows.

**Theorem 1.1.** Assume that **(A1)**-(**A3**) hold, and suppose  $\frac{1}{2} > \nu := \frac{\varepsilon + 2\beta \wedge \alpha \varepsilon^2}{2} + \alpha - 1 > 0$  and the initial value  $x \in \mathcal{D}(A)$ . Then, there exists some  $C = C(\nu, T) > 0$  such that

$$(1.10) \quad \int_0^T \mathbb{E} \|X_t - Y_t^{(n),\delta}\|_{\mathbb{H}}^2 dt \leq C \left\{ \delta^\nu + \frac{1}{\lambda_n^\nu} \right\}.$$

**Remark 1.1** If  $\beta \geq \frac{\alpha \varepsilon^2}{2}$ , then  $\nu > 0$  reduces to  $\varepsilon + \alpha \varepsilon^2 > 2(1 - \alpha)$  which is equivalent to  $\alpha > \frac{2-\varepsilon}{2+\varepsilon^2}$ . On the other hand, let  $Ax = \partial_\xi^2 x$  for  $x \in \mathcal{D}(A) := H^2(0, \pi) \cap H_0^1(0, \pi)$ . Then  $A$  is a self-adjoint negative operator and  $Ae_n = -n^2 e_n$ ,  $n \in \mathbb{N}$ , where  $e_n(\xi) = (2/\pi)^{1/2} \sin n\xi$  for  $\xi \in [0, \pi]$  and  $n \in \mathbb{N}$ . So,  $\lambda_n = n^2$  and we can take some  $\alpha \in (0, 1)$  such that (1.2) holds provided that  $\varepsilon > 0.732$ . Thus, taking  $\delta = \frac{T}{n}$  we obtain the convergence rate (in the sense of (1.10)) for EI scheme of stochastic heat equations driven by additive space-time white noise, where the drift  $b$  satisfies (1.3).

**Remark 1.2.** To avoid complicated computation, in the present setup we work only on the case that the drift is uniformly bounded. Nevertheless, employing the standard cut-off approach (see, e.g., [1]), we of course can extend our framework to the setting that the drift coefficient is unbounded.

The remainder of this paper is organized as follows: In Section 2, we investigate the regularities of the Kolmogorov equation which enables us to construct a transformation, which is a  $C^2$ -diffeomorphism and provides regular presentation of  $X_t$  and  $Y_t^{(n),\delta}$ , respectively; In Section 3, we focus on the spatial error and the temporal error so as to complete the proof of Theorem 1.1.

Convention: The letter  $c$  or  $C$  with or without subscripts will denote an unimportant constant, whose values may change in different places. Moreover, we use the shorthand notation  $a \lesssim b$  to mean  $a \leq cb$ . If the constant  $c$  depends on a parameter  $p$ , we shall also write  $c_p$  and  $a \lesssim_p b$ .

## 2 Regularity of Kolmogorov Equation

The proof of Theorem 1.1 relies heavily on the regularity of the following parabolic type partial differential equation, for fixed  $T > 0$ , any  $\lambda > 0$  and  $x \in \cup_{n=1}^{\infty} \mathbb{H}_n$ ,

$$(2.1) \quad \left( \partial_t u_t^\lambda + \nabla_{b_t} u_t^\lambda + b_t + \frac{1}{2} \sum_{i=1}^{\infty} \nabla_{e_i}^2 u_t^\lambda + \nabla_{A \cdot} u_t^\lambda \right)(x) = \lambda u_t^\lambda(x), \quad t \in [0, T], \quad u_T^\lambda = 0,$$

where  $\nabla_{e_i}^2 := \nabla_{e_i} \nabla_{e_i}$ , the second order gradient operator along the direction  $e_i$ . To characterize the regular property of the solution to (2.1), we consider the following O-U process

$$(2.2) \quad dZ_t^x = AZ_t^x dt + dW_t, \quad t \geq 0, \quad Z_0^x = x.$$

Under **(A1)**, it is well known that (2.2) has an up to modifications unique mild solution  $(Z_t^x)_{t \geq 0}$  (see, e.g., [4]) with the associated Markov semigroup  $(P_t^0)_{t \geq 0}$ . Consider the following integral equation

$$(2.3) \quad u_t^\lambda(x) = \int_t^T e^{-\lambda(s-t)} P_{s-t}^0 (\nabla_{b_s} u_s^\lambda + b_s)(x) ds, \quad x \in \mathbb{H}, \quad t \in [0, T].$$

It is well known that, for any  $x \in \cup_{n=1}^{\infty} \mathbb{H}_n$ , (2.1) and (2.3) are equivalent mutually.

Before we start the proof of Theorem 1.1, we prepare some auxiliary lemmas.

**Lemma 2.1.** Let **(A1)** and **(A2)** hold. Then,

- (1) There exists  $\lambda_T > 0$  such that, for any  $\lambda \geq \lambda_T$ , (2.3) has a unique solution  $u^\lambda \in C([0, T]; C_b^2(\mathbb{H}; \mathbb{H}))$ , which satisfies

$$(2.4) \quad \lim_{\lambda \rightarrow \infty} \{ \|u^\lambda\|_{T, \infty} + \|\nabla u^\lambda(-A)^\kappa\|_{T, \infty} + \|\nabla^2 u^\lambda\|_{T, \infty} \} = 0, \quad \kappa \in [0, 1/2].$$

(2) For any  $\theta \in [0, \alpha)$  and  $\lambda \geq \lambda_T$ ,

$$(2.5) \quad \sum_{i=1}^{\infty} \lambda_i^{\theta} \|\nabla_{e_i} u^{\lambda}\|_{T, \infty}^2 < \infty.$$

*Proof.* For the wellposedness of (2.3) and  $\lim_{\lambda \rightarrow \infty} \{\|u^{\lambda}\|_{T, \infty} + \|\nabla^2 u^{\lambda}\|_{T, \infty}\} = 0$  in (2.4), we refer to [20, Lemma 2.3] for more details. Hereinafter, we aim to show that

$$(2.6) \quad \lim_{\lambda \rightarrow \infty} \|\nabla u^{\lambda} (-A)^{\kappa}\|_{T, \infty} = 0, \quad \kappa \in [0, 1/2).$$

Note that the following Bismut formula

$$(2.7) \quad \nabla_{\eta} P_t^0 f(x) = \mathbb{E} \left( \frac{f(Z_t^x)}{t} \int_0^t \langle \nabla_{\eta} Z_s^x, dW_s \rangle \right), \quad t > 0, x, \eta \in \mathbb{H}, f \in \mathcal{B}_b(\mathbb{H}; \mathbb{H})$$

holds; see, e.g., [20, (2.8)] by using the Mallivin calculus. By Hölder's inequality and Itô's isometry, together with  $\nabla_{\eta} Z_t^x = e^{tA} \eta$ , we deduce that

$$(2.8) \quad \begin{aligned} \|\nabla_{(-A)^{\kappa} \eta} P_t^0 f(x)\|_{\mathbb{H}}^2 &\leq \frac{\mathbb{E} \|f(Z_t^x)\|_{\mathbb{H}}^2}{t^2} \int_0^t \|e^{sA} (-A)^{\kappa} \eta\|_{\mathbb{H}}^2 ds \\ &\lesssim \frac{P_t^0 \|f(x)\|_{\mathbb{H}}^2}{t^2} \sum_{j=1}^{\infty} \frac{(1 - e^{-2\lambda_j t}) \langle \eta, e_j \rangle^2}{\lambda_j^{1-2\kappa}}, \quad \kappa \in [0, 1/2), \end{aligned}$$

which, combining  $u^{\lambda} \in C([0, T]; C_b^1(\mathbb{H}; \mathbb{H}))$  with  $\|b\|_{T, \infty} < \infty$  and  $\lambda_j^{2\kappa-1} (1 - e^{-2\lambda_j t}) \lesssim t^{1-2\kappa}$ , yields that

$$\begin{aligned} \|\nabla_{(-A)^{\kappa} \eta} u_t^{\lambda}\|_{\mathbb{H}} &\leq \int_t^T e^{-\lambda(s-t)} \|\nabla_{(-A)^{\kappa} \eta} P_{s-t}^0 (\nabla_{b_s} u_s^{\lambda} + b_s)\|_{\mathbb{H}} ds \\ &\lesssim \|\eta\|_{\mathbb{H}} \int_0^T e^{-\lambda s} s^{-(\kappa + \frac{1}{2})} ds \\ &\lesssim \|\eta\|_{\mathbb{H}} \lambda^{\kappa - \frac{1}{2}}. \end{aligned}$$

Thereby, (2.6) follows immediately. Next, in (2.8), taking  $\kappa = 0$  and  $\eta = e_i$  gives that

$$(2.9) \quad \|\nabla_{e_i} P_t^0 f(x)\|_{\mathbb{H}}^2 \lesssim \frac{(1 - e^{-2\lambda_i t}) P_t^0 \|f(x)\|_{\mathbb{H}}^2}{\lambda_i t^2}.$$

This, in addition to  $u^{\lambda} \in C([0, T]; C_b^1(\mathbb{H}; \mathbb{H}))$  and  $\|b\|_{T, \infty} < \infty$ , leads to

$$(2.10) \quad \begin{aligned} \|\nabla_{e_i} u_t^{\lambda}\|_{\mathbb{H}} &\leq \int_t^T e^{-\lambda(s-t)} \|\nabla_{e_i} P_{s-t}^0 (\nabla_{b_s} u_s^{\lambda} + b_s)\|_{\mathbb{H}} ds \\ &\lesssim \int_0^T \frac{e^{-\lambda s} (1 - e^{-2\lambda_i s})^{\frac{1}{2}}}{\lambda_i^{\frac{1}{2}} s} ds \\ &\lesssim \frac{1}{\lambda_i^{\frac{1-\theta}{2}}} \int_0^T e^{-\lambda s} s^{\frac{\theta}{2}-1} ds \\ &\lesssim \frac{1}{\lambda_i^{\frac{1-\theta}{2}}}, \quad \theta \in (0, 1), \end{aligned}$$

where we have used the elementary inequality

$$(2.11) \quad |e^{-x} - e^{-y}| \leq c_\theta |x - y|^\theta, \quad x, y \geq 0, \quad \theta \in [0, 1]$$

for some constant  $c_\theta > 0$ . Hence, we deduce from (1.2) and (2.10) that

$$\sum_{i=1}^{\infty} \lambda_i^{\alpha-\theta} \|\nabla_{e_i} u^\lambda\|_{T,\infty}^2 \leq c_0 \sum_{i=1}^{\infty} \frac{1}{\lambda_i^{1-\alpha}} \leq c, \quad \theta \in (0, \alpha]$$

for some constants  $c_0, c > 0$ . As a result, (2.5) holds.  $\square$

**Remark 2.2.** In fact, all assertions in Lemma 2.1 hold under the assumptions **(A1)** and (1.6). Nevertheless, in Lemma 2.1 we write **(A2)** in lieu of (1.6) just to present in a consistent manner.

The Lemma below plays a crucial role in analyzing spatial error of numerical schemes.

**Lemma 2.3.** Let **(A1)** and **(A2)** hold and assume further  $\nu := \frac{\varepsilon+2\beta\wedge\alpha\varepsilon^2}{2} + \alpha - 1 > 0$ . Then, for any  $\lambda \geq \lambda_T$ ,

$$(2.12) \quad \sum_{i=1}^{\infty} \lambda_i^\nu \|\nabla_{e_i} \nabla_{e_i} u^\lambda\|_{T,\infty} < \infty.$$

*Proof.* From (2.7) and  $\nabla_\eta Z_t^x = e^{tA}\eta$ , we have

$$(2.13) \quad \frac{1}{2}(\nabla_{e_i} \nabla_\eta P_t^0 f)(x) = \frac{e^{-\frac{\lambda_i t}{2}}}{t} \mathbb{E} \left( (\nabla_{e_i} P_{t/2}^0 f)(Z_{t/2}^x) \int_0^{t/2} \langle e^{sA} \eta, dW_s \rangle \right).$$

By Hölder's inequality and Itô's isometry, it then follows from (2.9), (2.11) with  $\theta = 1$ , contractive property of  $e^{tA}$  and semigroup property of  $P_t^0$  that

$$(2.14) \quad \begin{aligned} \|(\nabla_{e_i} \nabla_\eta P_t^0 f)(x)\|_{\mathbb{H}}^2 &\lesssim \frac{e^{-\lambda_i t} \|\eta\|_{\mathbb{H}}^2}{t} \mathbb{E} \|(\nabla_{e_i} P_{t/2}^0 f)(Z_{t/2}^x)\|_{\mathbb{H}}^2 \\ &\lesssim \frac{e^{-\lambda_i t} \|\eta\|_{\mathbb{H}}^2}{t^2} P_t^0 \|f(x)\|_{\mathbb{H}}^2. \end{aligned}$$

Furthermore, Itô's isometry, (1.2) as well as (2.11) with  $\theta = \alpha$  yield that

$$(2.15) \quad \mathbb{E} \|Z_t^x - e^{tA}x\|_{\mathbb{H}}^2 = \int_0^t \|e^{(t-s)A}\|_{\text{HS}}^2 ds = \sum_{i=1}^{\infty} \frac{1 - e^{-2\lambda_i t}}{2\lambda_i} \lesssim t^\alpha.$$

For a mapping  $f : \mathbb{H} \rightarrow \mathbb{H}$  such that  $\|f(x) - f(y)\|_{\mathbb{H}} \lesssim \|x - y\|_{\mathbb{H}}^\varepsilon$ , let  $\tilde{f}_t(y) = f(y) - f(e^{tA}x)$ . Taking (2.14) and (2.15) into account and employing Jensen's inequality, we derive that

$$(2.16) \quad \begin{aligned} \|(\nabla_{e_i} \nabla_\eta P_t^0 f)(x)\|_{\mathbb{H}}^2 &= \|(\nabla_{e_i} \nabla_\eta P_t^0 \tilde{f})(x)\|_{\mathbb{H}}^2 \\ &\lesssim \frac{e^{-\lambda_i t} \|\eta\|_{\mathbb{H}}^2 \mathbb{E} \|Z_t^x - e^{tA}x\|_{\mathbb{H}}^{2\varepsilon}}{t^2} \\ &\lesssim \frac{e^{-\lambda_i t} \|\eta\|_{\mathbb{H}}^2}{t^{2-\alpha\varepsilon}}. \end{aligned}$$

For notational simplicity, set

$$(2.17) \quad \hat{f}_t^\lambda(x) := (\nabla_{b_t} u_t^\lambda + b_t)(x).$$

It is easy to see from (1.6) and (2.4) that

$$\|\hat{f}_t^\lambda(x) - \hat{f}_t^\lambda(y)\|_{\mathbb{H}} \lesssim \|x - y\|_{\mathbb{H}}^\varepsilon, \quad x, y \in \mathbb{H}.$$

Thus, combining (2.3) with (2.16) yields that

$$(2.18) \quad \begin{aligned} \|\nabla_{e_i} \nabla_\eta u_s^\lambda\|_{\mathbb{H}} &\leq \int_s^T e^{-\lambda(t-s)} \|\nabla_{e_i} \nabla_\eta P_{t-s}^0 \hat{f}_t^\lambda\|_{\mathbb{H}} dt \\ &\lesssim \|\eta\|_{\mathbb{H}} \int_0^T \frac{e^{-\lambda_i t/2}}{t^{1-\alpha\varepsilon/2}} dt \\ &\lesssim \frac{\|\eta\|_{\mathbb{H}}}{\lambda_i^{\alpha\varepsilon/2}}. \end{aligned}$$

For a mapping  $f : \mathbb{H} \rightarrow \mathbb{H}$  satisfying

$$(2.19) \quad \|f(x) - f(y)\|_{\mathbb{H}} \leq \sum_{i=1}^{\infty} \lambda_i^{-\beta_0} |x_i - y_i|^{\varepsilon_0}, \quad x, y \in \mathbb{H}$$

for some  $\beta_0 > 0$  and  $\varepsilon_0 \in (0, 1)$ , let

$$\tilde{f}_t^i(y) := f(y) - f(y + \langle e^{tA} x - y, e_i \rangle e_i), \quad x, y \in \mathbb{H}.$$

Now, (2.19), Jensen's inequality and Itô's isometry imply that

$$(2.20) \quad P_t^0 \|\tilde{f}_t^i(x)\|_{\mathbb{H}}^2 \leq \lambda_i^{-2\beta_0} \mathbb{E} |\Lambda_t^i|^{2\varepsilon_0} \lesssim \frac{(1 - e^{-2\lambda_i t})^{\varepsilon_0}}{\lambda_i^{\varepsilon_0 + 2\beta_0}},$$

where

$$\Lambda_t^i := \int_0^t \langle e^{(t-s)A} dW_s, e_i \rangle = \int_0^t e^{-\lambda_i(t-s)} d\beta_s^{(i)}.$$

In terms of (2.14) with  $\eta = e_i$  and (2.20), besides the notion of  $\tilde{f}_t^i$ , it follows from (2.11) that

$$(2.21) \quad \begin{aligned} \|(\nabla_{e_i} \nabla_{e_i} P_t^0 f)(x)\|_{\mathbb{H}}^2 &= \|(\nabla_{e_i} \nabla_{e_i} P_t^0 \tilde{f}_t^i)(x)\|_{\mathbb{H}}^2 \\ &\lesssim \frac{e^{-\lambda_i t} P_t^0 \|\tilde{f}_t^i(x)\|_{\mathbb{H}}^2}{t^2} \\ &\lesssim \frac{e^{-\lambda_i t}}{t^{2-\varepsilon_0\theta} \lambda_i^{\varepsilon_0(1-\theta)+2\beta_0}}, \quad \theta \in (0, 1]. \end{aligned}$$

Next, thanks to (1.5),  $\|b\|_{T,\infty} < \infty$  and (2.18), we obtain

$$(2.22) \quad \|\hat{f}_t^\lambda(x) - \hat{f}_t^\lambda(x + \langle y - x, e_i \rangle e_i)\|_{\mathbb{H}} \lesssim \lambda_i^{-(\beta \wedge \frac{\alpha\varepsilon^2}{2})} |x_i - y_i|^\varepsilon,$$

where  $\hat{f}_t^\lambda : \mathbb{H} \rightarrow \mathbb{H}$  is defined in (2.5), Hence, in light of (2.21) with  $\varepsilon_0 = \varepsilon$  and  $\beta_0 = \beta \wedge \frac{\alpha\varepsilon^2}{2}$  and (2.22), one infers that

$$\begin{aligned} \|\nabla_{e_i} \nabla_{e_i} u_s^\lambda\|_{\mathbb{H}} &\lesssim \frac{1}{\lambda_i^{\frac{\varepsilon(1-\theta)+2\beta\wedge\alpha\varepsilon^2}{2}}} \int_0^T \frac{e^{-\frac{\lambda_i t}{2}}}{t^{1-\frac{\varepsilon\theta}{2}}} dt \\ &\lesssim \frac{1}{\lambda_i^{\frac{\varepsilon+2\beta\wedge\alpha\varepsilon^2}{2}}}. \end{aligned}$$

This, along with (1.2), implies (2.12).  $\square$

The following lemma provides us with a regular representation of the continuous-time EI scheme (1.9).

**Lemma 2.4.** For any  $t \in [0, T]$  and  $\lambda \geq \lambda_T$ , it holds that

$$\begin{aligned} (2.23) \quad &Y_t^{(n),\delta} + u_t^\lambda(Y_t^{(n),\delta}) \\ &= e^{tA} \{x_n + u_0^\lambda(x_n)\} + \int_0^t e^{(t-s)A} (\lambda \mathbf{I} - A) u_s^\lambda(Y_s^{(n),\delta}) ds \\ &\quad + \int_0^t e^{(t-s)A} \{e^{(s-s_\delta)A} b_{s_\delta}^{(n)}(Y_{s_\delta}^{(n),\delta}) - b_s(Y_s^{(n),\delta})\} ds \\ &\quad + \int_0^t e^{(t-s)A} \nabla u_s^\lambda(Y_s^{(n),\delta}) (e^{(s-s_\delta)A} b_{s_\delta}^{(n)}(Y_{s_\delta}^{(n),\delta}) - b_s(Y_s^{(n),\delta})) ds \\ &\quad + \frac{1}{2} \sum_{i=1}^n \int_0^t e^{(t-s)A} (\nabla_{e^{(s-s_\delta)A} e_i}^2 u_s^\lambda - \nabla_{e_i}^2 u_s^\lambda)(Y_s^{(n),\delta}) ds \\ &\quad + \frac{1}{2} \sum_{i=n+1}^\infty \int_0^t e^{(t-s)A} (\nabla_{e_i}^2 u_s^\lambda)(Y_s^{(n),\delta}) ds \\ &\quad + \int_0^t e^{(t-s_\delta)A} dW_s^{(n)} + \int_0^t e^{(t-s)A} (\nabla_{e^{(s-s_\delta)A} dW_s^{(n)}} u_s^\lambda)(Y_s^{(n),\delta}), \end{aligned}$$

in which  $\mathbf{I}$  is the identity operator on  $\mathbb{H}$ .

*Proof.* Since  $A_n x = Ax$  for any  $x \in \mathbb{H}_n$ , (1.9) can be reformulated as

$$dY_t^{(n),\delta} = \{AY_t^{(n),\delta} + e^{(t-t_\delta)A} b_{t_\delta}^{(n)}(Y_{t_\delta}^{(n),\delta})\} dt + e^{(t-t_\delta)A} dW_t^{(n)}, \quad t > 0, \quad Y_0^{(n),\delta} = x_n.$$

Applying Itô's formula, for  $\lambda \geq \lambda_T$  we deduce from (2.1) that

$$\begin{aligned} &d\{Y_t^{(n),\delta} + u_t^\lambda(Y_t^{(n),\delta})\} \\ &= \left\{ AY_t^{(n),\delta} + e^{(t-t_\delta)A} b_{t_\delta}^{(n)}(Y_{t_\delta}^{(n),\delta}) + (\partial_t u_t^\lambda)(Y_t^{(n),\delta}) + (\nabla_A u_t^\lambda)(Y_t^{(n),\delta}) \right. \end{aligned}$$



$$\begin{aligned}
& + \left( \nabla_{e^{(t-t_\delta)A} b_{t_\delta}^{(n)}(Y_{t_\delta}^{(n),\delta})} u_t^\lambda \right) (Y_t^{(n),\delta}) + \frac{1}{2} \sum_{k=1}^n \left( \nabla_{e^{(t-t_\delta)A} e_k}^2 u_t^\lambda \right) (Y_t^{(n),\delta}) \Big\} dt \\
& + e^{(t-t_\delta)A} dW_t^{(n)} + \left( \nabla_{e^{(t-t_\delta)A} dW_t^{(n)}} u_t^\lambda \right) (Y_t^{(n),\delta}) \\
& = \left\{ AY_t^{(n),\delta} + \lambda u_t^\lambda(Y_t^{(n),\delta}) + e^{(t-t_\delta)A} b_{t_\delta}^{(n)}(Y_{t_\delta}^{(n),\delta}) - b_t(Y_t^{(n),\delta}) \right. \\
& \quad + \nabla u_t^\lambda(Y_t^{(n),\delta}) \left( e^{(t-t_\delta)A} b_{t_\delta}^{(n)}(Y_{t_\delta}^{(n),\delta}) - b_t(Y_t^{(n),\delta}) \right) + \frac{1}{2} \sum_{k=1}^n \left( \nabla_{e^{(t-t_\delta)A} e_k}^2 u_t^\lambda \right) (Y_t^{(n),\delta}) \\
& \quad \left. - \frac{1}{2} \sum_{k=1}^\infty (\nabla_{e_k}^2 u_t^\lambda)(Y_t^{(n),\delta}) \right\} dt + e^{(t-t_\delta)A} dW_t^{(n)} + \left( \nabla_{e^{(t-t_\delta)A} dW_t^{(n)}} u_t^\lambda \right) (Y_t^{(n),\delta}).
\end{aligned}$$

This, in addition to integration by parts, further implies that

$$\begin{aligned}
& \int_0^t A e^{(t-s)A} \{Y_s^{(n),\delta} + u_s^\lambda(Y_s^{(n),\delta})\} ds \\
& = -e^{(t-s)A} \{Y_s^{(n),\delta} + u_s^\lambda(Y_s^{(n),\delta})\} \Big|_0^t + \int_0^t e^{(t-s)A} d\{Y_s^{(n),\delta} + u_s^\lambda(Y_s^{(n),\delta})\} \\
& = -\{Y_t^{(n),\delta} + u_t^\lambda(Y_t^{(n),\delta})\} + e^{tA} \{x_n + u_0^\lambda(x_n)\} + \int_0^t A e^{(t-r)A} Y_r^{(n),\delta} dr \\
& \quad + \lambda \int_0^t e^{(t-r)A} u_r^\lambda(Y_r^{(n),\delta}) dr + \int_0^t e^{(t-r)A} \{e^{(r-r_\delta)A} b_{r_\delta}^{(n)}(Y_{r_\delta}^{(n),\delta}) - b_r(Y_r^{(n),\delta})\} dr \\
& \quad + \int_0^t e^{(t-r)A} \nabla u_r^\lambda(Y_r^{(n),\delta}) \left( e^{(r-r_\delta)A} b_{r_\delta}^{(n)}(Y_{r_\delta}^{(n),\delta}) - b_r(Y_r^{(n),\delta}) \right) dr \\
& \quad + \frac{1}{2} \sum_{k=1}^n \int_0^t e^{(t-r)A} \left( \nabla_{e^{(r-r_\delta)A} e_k}^2 u_r^\lambda \right) (Y_r^{(n),\delta}) dr - \frac{1}{2} \sum_{k=1}^\infty \int_0^t e^{(t-r)A} (\nabla_{e_k}^2 u_r^\lambda)(Y_r^{(n),\delta}) dr \\
& \quad + \int_0^t e^{(t-r_\delta)A} dW_r^{(n)} + \int_0^t e^{(t-r)A} \left( \nabla_{e^{(r-r_\delta)A} dW_r^{(n)}} u_r^\lambda \right) (Y_r^{(n),\delta}).
\end{aligned}$$

As a consequence, the desired assertion (2.23) is now available.  $\square$

The following lemma concerns the continuity in the mean  $L^2$ -norm sense for the displacement of  $(Y_t^{(n),\delta})_{t \in [0, T]}$ .

**Lemma 2.5.** Let **(A1)** and **(A2)** hold and assume that the inial value  $x \in \mathcal{D}(A)$ . Then,

$$(2.24) \quad \sup_{t \in [0, T]} \mathbb{E} \|Y_t^{(n),\delta} - Y_{t_\delta}^{(n),\delta}\|_{\mathbb{H}}^2 \lesssim_T \delta^\alpha.$$

*Proof.* To make the content self-contained, we here give a sketch although the corresponding argument of (2.24) is quite standard. By virtue of (1.9), it follows immediately that

$$Y_t^{(n),\delta} - Y_{t_\delta}^{(n),\delta} = (e^{(t-t_\delta)A} - \mathbf{I}) e^{t_\delta A} x_n$$

$$\begin{aligned}
& + \int_0^{t_\delta} (e^{(t-t_\delta)A} - \mathbf{I})e^{(t_\delta-s_\delta)A}b_{s_\delta}^{(n)}(Y_{s_\delta}^{(n),\delta})ds \\
& + \int_0^{t_\delta} (e^{(t-t_\delta)A} - \mathbf{I})e^{(t_\delta-s_\delta)A}dW_s^{(n)} \\
& + \int_{t_\delta}^t e^{(t-s_\delta)A}b_{s_\delta}^{(n)}(Y_{s_\delta}^{(n),\delta})ds + \int_{t_\delta}^t e^{(t-s_\delta)A}dW_s^{(n)}.
\end{aligned}$$

Recall the elementary inequalities: for any  $\gamma \in (0, 1]$ ,

$$(2.25) \quad \|(-A)^\gamma e^{tA}\| \leq t^{-\gamma} \quad \text{and} \quad \|(-A)^{-\gamma}(e^{tA} - \mathbf{I})\| \leq t^\gamma.$$

Next, according to Hölder's inequality and Itô's isometry and by taking contractive property of  $e^{tA}$  and  $\|b\|_{T,\infty} < \infty$  into account, we derive from (2.25) that

$$\begin{aligned}
\mathbb{E}\|Y_t^{(n),\delta} - Y_{t_\delta}^{(n),\delta}\|_{\mathbb{H}}^2 & \lesssim_T \|(e^{(t-t_\delta)A} - \mathbf{I})e^{t_\delta A}x_n\|_{\mathbb{H}}^2 \\
& + \int_0^{t_\delta} \mathbb{E}\|(e^{(t-t_\delta)A} - \mathbf{I})e^{(t_\delta-s_\delta)A}b_{s_\delta}^{(n)}(Y_{s_\delta}^{(n),\delta})\|_{\mathbb{H}}^2 ds \\
& + \int_0^{t_\delta} \|(e^{(t-t_\delta)A} - \mathbf{I})e^{(t_\delta-s_\delta)A}\|_{\mathcal{L}_2}^2 ds \\
& + \delta \int_{t_\delta}^t \mathbb{E}\|e^{(t-s_\delta)A}b_{s_\delta}^{(n)}(Y_{s_\delta}^{(n),\delta})\|_{\mathbb{H}}^2 ds + \int_{t_\delta}^t \|e^{(t-s_\delta)A}\|_{\mathcal{L}_2}^2 ds \\
& \lesssim_T \|(e^{(t-t_\delta)A} - \mathbf{I})(-A)^{-1}\|^2 \|Ax\|_{\mathbb{H}}^2 \\
& + \int_0^{t_\delta} \|(e^{(t-t_\delta)A} - \mathbf{I})(-A)^{-\alpha/2}\|^2 \|(-A)^{\alpha/2}e^{(t_\delta-s_\delta)A}\|^2 \mathbb{E}\|b_{s_\delta}^{(n)}(Y_{s_\delta}^{(n),\delta})\|_{\mathbb{H}}^2 ds \\
& + \int_0^{t_\delta} \|(e^{(t-t_\delta)A} - \mathbf{I})(-A)^{-\alpha/2}\|^2 \|(-A)^{\alpha/2}e^{(t_\delta-s_\delta)A}\|_{\mathcal{L}_2}^2 ds \\
& + \delta \int_{t_\delta}^t \mathbb{E}\|b_{s_\delta}^{(n)}(Y_{s_\delta}^{(n),\delta})\|_{\mathbb{H}}^2 ds + \int_{t_\delta}^t \|e^{(t-s_\delta)A}\|_{\mathcal{L}_2}^2 ds \\
& \lesssim_T \delta^2 + \delta^\alpha \int_0^{t_\delta} (t_\delta - s_\delta)^{-\alpha} ds + \delta^\alpha \sum_{i=1}^{\infty} \lambda_i^\alpha \int_0^{t_\delta} e^{-2\lambda_i(t_\delta-s_\delta)} ds \\
& + \sum_{i=1}^{\infty} \int_0^{t-t_\delta} e^{-2\lambda_i s} ds \\
& \lesssim_T \delta^2 + \delta^\alpha \int_0^{t_\delta} s^{-\alpha} ds + \delta^\alpha \sum_{i=1}^{\infty} \lambda_i^\alpha \int_0^{t_\delta} e^{-2\lambda_i s} ds + \sum_{i=1}^{\infty} \int_0^{t-t_\delta} e^{-2\lambda_i s} ds \\
& \lesssim_T \delta^\alpha + \delta^\alpha \sum_{i=1}^{\infty} \frac{1}{\lambda_i^{1-\alpha}} + \sum_{i=1}^{\infty} \frac{1 - e^{-2\lambda_i(t-t_\delta)}}{\lambda_i} \\
& \lesssim_T \delta^\alpha + \delta^\alpha \sum_{i=1}^{\infty} \frac{1}{\lambda_i^{1-\alpha}}
\end{aligned}$$

$$\lesssim_T \delta^\alpha,$$

where in the penultimate display we have used (2.11) with  $\theta = \alpha$  and in the last step utilized (1.2).  $\square$

### 3 Proof of Theorem 1.1

With Lemmas 2.1-2.5 in hand, we now in a position to complete the proof of Theorem 1.1. In view of (2.4), we deduce that there exists  $\hat{\lambda}_T \geq \lambda_T$  such that for any  $\lambda \geq \hat{\lambda}_T$

$$(3.1) \quad \|u^\lambda\|_{T,\infty} + \|\nabla u^\lambda\|_{T,\infty} + \|\nabla u^\lambda(-A)^\kappa\|_{T,\infty} + \|\nabla^2 u^\lambda\|_{T,\infty} \leq \frac{1}{3\sqrt{2}}.$$

In what follows, we shall fix  $\lambda \geq \hat{\lambda}_T$  so that (3.1) holds. For notational simplicity, set  $\theta_t^\lambda(x) := x + u_t^\lambda(x)$ ,  $x \in \mathbb{H}$ . Since

$$\begin{aligned} \|X_t - Y_t^{(n),\delta}\|_{\mathbb{H}}^2 &\leq 2\|\theta_t^\lambda(X_t) - \theta_t^\lambda(Y_t^{(n),\delta})\|_{\mathbb{H}}^2 + 2\|u_t^\lambda(X_t) - u_t^\lambda(Y_t^{(n),\delta})\|_{\mathbb{H}}^2 \\ &\leq 2\|\theta_t^\lambda(X_t) - \theta_t^\lambda(Y_t^{(n),\delta})\|_{\mathbb{H}}^2 + \frac{1}{9}\|X_t - Y_t^{(n),\delta}\|_{\mathbb{H}}^2, \end{aligned}$$

we have

$$(3.2) \quad \Gamma_t^{(n),\delta} := \mathbb{E}\|X_t - Y_t^{(n),\delta}\|_{\mathbb{H}}^2 \leq \frac{9}{4}\mathbb{E}\|\theta_t^\lambda(X_t) - \theta_t^\lambda(Y_t^{(n),\delta})\|_{\mathbb{H}}^2.$$

According to [20, Proposition 2.5], one has

$$\begin{aligned} (3.3) \quad \theta_t^\lambda(X_t) &= e^{tA}\theta_0^\lambda(x) + \int_0^t \{(\lambda\mathbf{I} - A)e^{(t-s)A}u_s^\lambda(X_s)\}ds \\ &\quad + \int_0^t e^{(t-s)A}\{dW_s + (\nabla_{dW_s}u_s^\lambda)(X_s)\}. \end{aligned}$$

In view of (2.23) and (3.3), we find that

$$\begin{aligned}
\Gamma_t^{(n),\delta} &\leq 9 \left\{ \|e^{tA}(x - x_n)\|_{\mathbb{H}}^2 + \|e^{tA}(u_0^\lambda(x) - u_0^\lambda(x_n))\|_{\mathbb{H}}^2 \right. \\
&\quad + \mathbb{E} \left\| \int_0^t e^{(t-s)A} (\lambda \mathbf{I} - A) \{u_s^\lambda(X_s) - u_s^\lambda(Y_s^{(n),\delta})\} ds \right\|_{\mathbb{H}}^2 \\
&\quad + \mathbb{E} \left\| \int_0^t e^{(t-s)A} \{e^{(s-s_\delta)A} b_{s_\delta}^{(n)}(Y_{s_\delta}^{(n),\delta}) - b_s(Y_s^{(n),\delta})\} ds \right\|_{\mathbb{H}}^2 \\
&\quad + \mathbb{E} \left\| \int_0^t e^{(t-s)A} \nabla u_s^\lambda(Y_s^{(n),\delta}) (e^{(s-s_\delta)A} b_{s_\delta}^{(n)}(Y_{s_\delta}^{(n),\delta}) - b_s(Y_s^{(n),\delta})) ds \right\|_{\mathbb{H}}^2 \\
(3.4) \quad &\quad + \mathbb{E} \left\| \sum_{i=1}^n \int_0^t e^{(t-s)A} \left( \nabla_{e^{(s-s_\delta)A} e_i}^2 u_s^\lambda - \nabla_{e_i}^2 u_s^\lambda \right) (Y_s^{(n),\delta}) ds \right\|_{\mathbb{H}}^2 \\
&\quad + \mathbb{E} \left\| \sum_{i=n+1}^\infty \int_0^t e^{(t-s)A} \left( \nabla_{e_i}^2 u_s^\lambda \right) (Y_s^{(n),\delta}) ds \right\|_{\mathbb{H}}^2 \\
&\quad + \mathbb{E} \left\| \int_0^t e^{(t-s)A} dW_s - \int_0^t e^{(t-s_\delta)A} dW_s^{(n)} \right\|_{\mathbb{H}}^2 \\
&\quad + \mathbb{E} \left\| \int_0^t e^{(t-s)A} \left( \nabla_{dW_s} u_s^\lambda \right) (X_s) - \int_0^t e^{(t-s)A} \left( \nabla_{e^{(s-s_\delta)A} dW_s^{(n)}} u_s^\lambda \right) (Y_s^{(n),\delta}) \right\|_{\mathbb{H}}^2 \Big\} \\
&=: 9(\Lambda_t^{(1)} + \Lambda_t^{(2)} + \dots + \Lambda_t^{(9)}).
\end{aligned}$$

Using Hölder's inequality and Fubini's theorem, we deduce from (3.1) that

$$\begin{aligned}
\int_0^t e^{-2\lambda s} \Lambda_s^{(3)} ds &= \sum_{i=1}^\infty \mathbb{E} \int_0^t e^{-2\lambda s} \left( \int_0^s \langle e^{(s-r)A} (\lambda \mathbf{I} - A) \{u_r^\lambda(X_r) - u_r^\lambda(Y_r^{(n),\delta})\}, e_i \rangle dr \right)^2 ds \\
&= \sum_{i=1}^\infty \mathbb{E} \int_0^t \left( \int_0^s e^{-\lambda_i(s-r)-\lambda s} (\lambda + \lambda_i) \langle u_r^\lambda(X_r) - u_r^\lambda(Y_r^{(n),\delta}) \rangle, e_i \rangle dr \right)^2 ds \\
&\leq \sum_{i=1}^\infty \int_0^t \left( \int_0^s e^{-(\lambda_i+\lambda)(s-r)} (\lambda + \lambda_i) dr \right. \\
&\quad \times \left. \int_0^s e^{-(\lambda+\lambda_i)(s-r)-2\lambda r} (\lambda + \lambda_i) \mathbb{E} \langle u_r^\lambda(X_r) - u_r^\lambda(Y_r^{(n),\delta}) \rangle, e_i \rangle^2 dr \right) ds \\
&\leq \sum_{i=1}^\infty \int_0^t \int_0^s e^{-(\lambda+\lambda_i)(s-r)-2\lambda r} (\lambda + \lambda_i) \mathbb{E} \langle u_r^\lambda(X_r) - u_r^\lambda(Y_r^{(n),\delta}) \rangle, e_i \rangle^2 dr ds \\
&= \sum_{i=1}^\infty \int_0^t \left( \int_r^t e^{-(\lambda+\lambda_i)(s-r)} (\lambda + \lambda_i) ds \right) e^{-2\lambda r} \mathbb{E} \langle u_r^\lambda(X_r) - u_r^\lambda(Y_r^{(n),\delta}) \rangle, e_i \rangle^2 dr \\
&\leq \sum_{i=1}^\infty \int_0^t e^{-2\lambda s} \mathbb{E} \langle u_s^\lambda(X_s) - u_s^\lambda(Y_s^{(n),\delta}) \rangle, e_i \rangle^2 ds \\
&= \int_0^t e^{-2\lambda s} \mathbb{E} \|u_s^\lambda(X_s) - u_s^\lambda(Y_s^{(n),\delta})\|_{\mathbb{H}}^2 ds
\end{aligned}$$

$$\leq \frac{1}{18} \int_0^t e^{-2\lambda s} \Gamma_s^{(n),\delta} ds,$$

which, combining with (3.4), further leads to

$$\int_0^t e^{-2\lambda s} \Gamma_s^{(n),\delta} ds \leq \frac{1}{2} \int_0^t e^{-2\lambda s} \Gamma_s^{(n),\delta} ds + 9 \sum_{i=1, i \neq 3}^9 \int_0^t e^{-2\lambda s} \Lambda_s^{(i)} ds.$$

We therefore obtain that

$$(3.5) \quad \int_0^t e^{-2\lambda s} \Gamma_s^{(n),\delta} ds \leq 18 \sum_{i=1, i \neq 3}^9 \int_0^t e^{-2\lambda s} \Lambda_s^{(i)} ds.$$

To achieve the desired assertion (1.10), in the sequel, we aim to estimate the terms  $\Lambda_t^{(i)}$ ,  $i \neq 3$ , step-by-step. By the contraction property of  $e^{tA}$  and thanks to (3.1) and  $x \in \mathcal{D}(A)$ ,

$$\Lambda_t^{(1)} + \Lambda_t^{(2)} \lesssim \|x - x_n\|_{\mathbb{H}}^2 \lesssim \frac{1}{\lambda_n^2} \sum_{i=n+1}^{\infty} \lambda_i^2 \langle x, e_i \rangle^2 \lesssim \frac{\|Ax\|_{\mathbb{H}}^2}{\lambda_n^2} \lesssim \frac{1}{\lambda_n^2}.$$

Taking Hölder's inequality and (2.25) with  $\gamma = \alpha\varepsilon/2$  into consideration and taking advantage of contractive property of  $e^{tA}$  and  $\|b\|_{T,\infty} < \infty$  as well as **(A3)**, we deduce from  $\alpha\varepsilon \in (0, 1)$  that

$$\begin{aligned} \Lambda_t^{(4)} &\lesssim_T \int_0^t \mathbb{E} \|b_{s_\delta}(Y_{s_\delta}^{(n),\delta}) - b_s(Y_s^{(n),\delta})\|_{\mathbb{H}}^2 ds + \int_0^t \mathbb{E} \|b_s(Y_{s_\delta}^{(n),\delta}) - b_s(Y_s^{(n),\delta})\|_{\mathbb{H}}^2 ds \\ &\quad + \int_0^t \mathbb{E} \|e^{(t-s_\delta)A} \{\pi_n - \mathbf{I}\} b_s(Y_s^{(n),\delta})\|_{\mathbb{H}}^2 ds + \int_0^t \mathbb{E} \|e^{(t-s)A} \{e^{(s-s_\delta)A} - \mathbf{I}\} b_s(Y_s^{(n),\delta})\|_{\mathbb{H}}^2 ds \\ &\lesssim_T \delta^{\alpha\varepsilon} + \sum_{i=n+1}^{\infty} \int_0^t e^{-2\lambda_i(t-s_\delta)} \mathbb{E} \langle b_s(Y_s^{(n),\delta}), e_i \rangle^2 ds \\ &\quad + \int_0^t \|e^{(t-s)A} (-A)^{\alpha\varepsilon/2}\|^2 \|(-A)^{-\alpha\varepsilon/2} \{e^{(s-s_\delta)A} - \mathbf{I}\}\|^2 \mathbb{E} \|b_s(Y_s^{(n),\delta})\|_{\mathbb{H}}^2 ds \\ &\lesssim_T \delta^{\alpha\varepsilon} + \int_0^t e^{-2\lambda_n(t-s_\delta)} \mathbb{E} \|b_s(Y_s^{(n),\delta})\|_{\mathbb{H}}^2 ds + \delta^{\alpha\varepsilon} \int_0^t (t-s)^{-\alpha\varepsilon} \mathbb{E} \|b_s(Y_s^{(n),\delta})\|_{\mathbb{H}}^2 ds \\ &\lesssim_T \delta^{\alpha\varepsilon} + \frac{1}{\lambda_n}. \end{aligned}$$

By the aid of Jensen's inequality, in addition to (1.6), (2.4), (2.24), (2.25), (3.1), contractive property of  $e^{tA}$  and  $\|b\|_{T,\infty} < \infty$  along with **(A3)** yield that

$$\begin{aligned} \Lambda_t^{(5)} &\lesssim \int_0^t \mathbb{E} \|b_{s_\delta}(Y_{s_\delta}^{(n),\delta}) - b_{s_\delta}(Y_s^{(n),\delta})\|_{\mathbb{H}}^2 ds \\ &\quad + \int_0^t \mathbb{E} \|b_{s_\delta}(Y_s^{(n),\delta}) - b_s(Y_s^{(n),\delta})\|_{\mathbb{H}}^2 ds \end{aligned}$$

$$\begin{aligned}
& + \int_0^t \mathbb{E} \|\nabla u_s^\lambda(Y_s^{(n),\delta})(-A)^{\alpha\varepsilon/2}\|^2 \|(-A)^{-\alpha\varepsilon/2}(e^{(s-s_\delta)A} - \mathbf{I})\|^2 ds \\
& + \int_0^t \mathbb{E} \|\nabla u_s^\lambda(Y_s^{(n),\delta})(-A)^\nu\|^2 \|(-A)^{-\nu}(\pi_n - \mathbf{I})\|^2 ds \\
& \lesssim \delta^{\alpha\varepsilon} + \frac{1}{\lambda_n^{2\nu}},
\end{aligned}$$

where we have also used  $\frac{1}{2} > \nu = \frac{\varepsilon + 2\beta \wedge \alpha \varepsilon^2}{2} + \alpha - 1 > 0$ . We thus obtain from (2.12) that

$$\begin{aligned}
\Lambda_t^{(6)} & \lesssim \int_0^t \mathbb{E} \left( \sum_{i=1}^n (1 - e^{-\lambda_i(s-s_\delta)})^2 \|(\nabla_{e_i}^2 u_s^\lambda)(Y_s^{(n),\delta})\|_{\mathbb{H}} \right)^2 ds \\
(3.6) \quad & \lesssim \delta^{2\nu} \int_0^t \mathbb{E} \left( \sum_{i=1}^n \lambda_i^\nu \|(\nabla_{e_i}^2 u_s^\lambda)(Y_s^{(n),\delta})\|_{\mathbb{H}} \right)^2 ds \\
& \lesssim \delta^{2\nu},
\end{aligned}$$

where we have used  $\sup_{x>0} \{(1 - e^{-x})x^{-\nu}\} < \infty$  for  $\nu \in (0, 1)$ . Also, with the help of (2.12),

$$\begin{aligned}
\Lambda_t^{(7)} & \lesssim \int_0^t \mathbb{E} \left( \sum_{i=n+1}^\infty \|(\nabla_{e_i}^2 u_s^\lambda)(Y_s^{(n),\delta})\|_{\mathbb{H}} \right)^2 ds \\
(3.7) \quad & \lesssim \frac{1}{\lambda_n^{2\nu}} \int_0^t \mathbb{E} \left( \sum_{i=n+1}^\infty \lambda_i^\nu \|(\nabla_{e_i}^2 u_s^\lambda)(Y_s^{(n),\delta})\|_{\mathbb{H}} \right)^2 ds \\
& \lesssim \frac{1}{\lambda_n^{2\nu}}.
\end{aligned}$$

Employing Itô's isometry, (1.2) and (2.11) with  $\theta = \alpha/2$ , we get that

$$\begin{aligned}
\Lambda_t^{(8)} & \lesssim \sum_{i=n+1}^\infty \int_0^t e^{-2\lambda_i(t-s)} ds + \sum_{i=1}^n \int_0^t e^{-2\lambda_i(t-s)} (1 - e^{-\lambda_i(s-s_\delta)})^2 ds \\
& \lesssim \sum_{i=n+1}^\infty \int_0^t e^{-2\lambda_i s} ds + \delta^\alpha \sum_{i=1}^n \lambda_i^\alpha \int_0^t e^{-2\lambda_i s} ds \\
& \lesssim \sum_{i=n+1}^\infty \frac{1}{\lambda_i} + \delta^\alpha \sum_{i=1}^n \frac{1}{\lambda_i^{1-\alpha}} \\
& \lesssim \frac{1}{\lambda_n^\alpha} + \delta^\alpha.
\end{aligned}$$

Let

$$\Theta_t = \int_0^t \|e^{(t-r)A}\|_{\text{HS}}^2 \mathbb{E} \|(\nabla u_r^\lambda)(X_r) - (\nabla u_r^\lambda)(Y_r^{(n),\delta})\|^2 dr.$$

Since  $0 < \nu < \frac{1}{2} \wedge \alpha$ , again, an application of Itô's isometry, together with (2.11) with  $\theta = \nu/2$ , implies that

$$\Lambda_t^{(9)} \leq c_1 \left\{ \Theta_t + \sum_{i=1}^n \int_0^t (1 - e^{-\lambda_i(r-r_\delta)})^2 \mathbb{E} \|(\nabla_{e_i} u_r^\lambda)(Y_r^{(n),\delta})\|_{\mathbb{H}}^2 dr \right\}$$

$$\begin{aligned}
& + \sum_{i=n+1}^{\infty} \int_0^t \mathbb{E} \|(\nabla_{e_i} u_r^\lambda)(X_r)\|_{\mathbb{H}}^2 dr \Big\} \\
& \leq c_2 \left\{ \Theta_t + \delta^\nu \sum_{i=1}^n \lambda_i^\nu \int_0^t \mathbb{E} \|(\nabla_{e_i} u_r^\lambda)(Y_r^{(n),\delta})\|_{\mathbb{H}}^2 dr \right. \\
& \quad \left. + \lambda_n^{-\nu} \sum_{i=n+1}^{\infty} \lambda_i^\nu \int_0^t \mathbb{E} \|(\nabla_{e_i} u_r^\lambda)(X_r)\|_{\mathbb{H}}^2 dr \right\} \\
& \leq c_3 \{ \Theta_t + \delta^\nu + \lambda_n^{-\nu} \}
\end{aligned}$$

for some constants  $c_1, c_2, c_3 > 0$ , where we have used (2.5) with  $\theta = \nu$  in the last procedure. By means of Hölder's inequality, (1.2) and (2.11) with  $\theta = \frac{\alpha}{2}$ , we find that

$$\begin{aligned}
(3.8) \quad \int_0^t e^{-2\lambda s} \|e^{sA}\|_{\text{HS}}^2 ds &= \sum_{i=1}^{\infty} \int_0^t e^{-2\lambda s} e^{-2\lambda_i s} ds \\
&\leq \sum_{i=1}^{\infty} \left( \int_0^t e^{-\frac{(2-\alpha)\lambda_i s}{1-\alpha}} ds \right)^{\frac{2(1-\alpha)}{2-\alpha}} \left( \int_0^t e^{-\frac{2(2-\alpha)\lambda s}{\alpha}} ds \right)^{\frac{\alpha}{2-\alpha}} \\
&= \sum_{i=1}^{\infty} \left( \frac{1 - e^{-\frac{(2-\alpha)\lambda_i t}{1-\alpha}}}{\frac{(2-\alpha)\lambda_i}{1-\alpha}} \right)^{\frac{2(1-\alpha)}{2-\alpha}} \left( \frac{\alpha}{2(2-\alpha)\lambda} \right)^{\frac{\alpha}{2-\alpha}} \\
&\leq c_4 \lambda^{-\frac{\alpha}{2-\alpha}}
\end{aligned}$$

for some constant  $c_4 > 0$ . Next, via Fubini's theorem, (3.1) and (3.8), we deduce that

$$\begin{aligned}
\int_0^t e^{-2\lambda s} \Theta_s ds &= \int_0^t e^{-2\lambda r} \mathbb{E} \|(\nabla u_r^\lambda)(X_r) - (\nabla u_r^\lambda)(Y_r^{(n),\delta})\|^2 \left( \int_0^{t-r} e^{-2\lambda s} \|e^{sA}\|_{\text{HS}}^2 ds \right) dr \\
&\leq c_4 \lambda^{-\frac{\alpha}{2-\alpha}} \int_0^t e^{-2\lambda r} \mathbb{E} \|(\nabla u_r^\lambda)(X_r) - (\nabla u_r^\lambda)(Y_r^{(n),\delta})\|^2 dr \\
&\leq \frac{c_4}{18\lambda^{\frac{\alpha}{2-\alpha}}} \int_0^t e^{-2\lambda s} \Gamma_s^{(n),\delta} ds.
\end{aligned}$$

Furthermore, taking  $\lambda \geq \hat{\lambda}_T$  such that  $\frac{c_3 c_4}{\lambda^{\frac{\alpha}{2-\alpha}}} < 1$  and combining all the estimates above with (3.5), we arrive at

$$\int_0^t e^{-2\lambda s} \Gamma_s^{(n),\delta} ds \lesssim_T \delta^\nu + \frac{1}{\lambda_n^\nu},$$

where we have also utilized  $\nu \in (0, \alpha)$  and  $\nu \in (0, \alpha\varepsilon)$ . This therefore implies the desired assertion.

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